proof of Interior Schauder estimate!!

By Lemma 4.5.

$$[u]_{2,\alpha;B_{1/2}(x_0)} \le \delta C_1 \left(\|f\|_{C^\alpha(B_1(x_0))} + [u]_{0;B_1(x_0)} + [u]_{2;B_1(x_0)} \right)$$

$\forall x_0 \in B_{3/2}(0)$.

$$\Rightarrow [u]_{2,\alpha;B_1(0)} \le C \sup_{|x_0| \le 1/2} [u]_{2,\alpha;B_{1/2}(x_0)}$$

$$\le C\left(\|f\|_{C^\alpha(B_2)} + [u]_{0;B_2} + [u]_{2;B_{3/2}} \right)$$

Define $Q(x) = (2-4|x|) \max\limits_{|\gamma|=2} |D^\gamma u(x)|$

observe $\sup Q \ge \frac{1}{2} \max\limits_{|\gamma|=2} |D^\gamma u(x)| \quad \forall |x| \le \frac{3}{2}$,

$\Rightarrow [u]_{2;B_{3/2}(0)} \le 2 \sup Q$.

we only need to show $\sup Q \le C(\|f\|_{C^\alpha} + [u]_0)$

since $[u]_{2;B_2} < +\infty$.

$\Rightarrow Q \to 0$ as $\|x\| \to 2$

$\exists\, x_0 \in B_2$. s.t. $Q(x_0) = \max Q$

Let $\rho = \frac{1}{3}(2 - \|x_0\|) < 1$

$$\Rightarrow Q(x_0) = 9\rho^2 \sup_{|\gamma|=2} |D^\gamma u(x_0)|$$

$$\leq 9\rho^2 [u]_{2;B_\rho(x_0)} = 9[\hat{u}]_{2;B_1(0)}$$

where $\hat{u}(x) = u(x_0 + \rho x)$.

By the Interpolation inequality

given ε. $\exists\, C_\varepsilon$ s.t.

$$[\hat{u}]_{2;B_1(0)} \leq \varepsilon [\hat{u}]_{2,\alpha;B_1} + C_\varepsilon [\hat{u}]_{0;B_1}$$

By Lemma 5. and $\rho \leq 1$

$$[\hat{u}]_{2,\alpha;B_1} \leq C_1 \Big(\|f\|_{C^\alpha(B_{2\rho}(x_0))} + [u]_{0;B_{2\rho}(x_0)} + \rho^2 [u]_{2;B_{2\rho}(x_0)} \Big)$$

we combine all inequalities above.

$$\frac{1}{9}Q(x_0) \le [\hat{u}]_{2;B_1(0)}$$

$$\le \varepsilon C_1 \|f\|_{C^\alpha(B_2)} + (C_2 + \varepsilon C_1)[u]_{0;B_2}$$

$$+ \varepsilon C_1 (\max Q) \xrightarrow{} Q(x_0)$$

$$\left(\begin{array}{l} \because \text{ if } y \in B_{2\rho}(x_0), \text{ then } 2 - \|y\| \ge \rho \\ \Rightarrow \rho^2 \sup_{|\gamma|=2} |D^\gamma u(y)| \le Q(y) \quad \forall y \in B_{2\rho}(x_0) \\ \Rightarrow \rho^2 [u]_{2;B_{2\rho}(x_0)} \le \max Q. \end{array} \right)$$

We choose $\varepsilon = \min\left\{1, \frac{1}{18 C_1}\right\}$

$$\Rightarrow \varepsilon C_1 \max Q \le \frac{1}{18} Q(x_0)$$

$$\therefore \frac{1}{18} Q(x_0) = \left(\frac{1}{9} - \frac{1}{18}\right) Q(x_0)$$

$$\le C\left(\|f\|_{C^\alpha(B_2)} + [u]_{0;B_2}\right)$$

Interior + Boundary schauder = Global schauder

↳ 3+4. ↳ pset 4 + Final. ↳ Lecture 2

Def) We say that $\Omega \subset \mathbb{R}^n$ satisfies the exterior sphere condition if given $x \in \partial\Omega$, there exists a ball $B_r(y) \cap \Omega = \phi$ such that $x \in \partial B_r(y)$.

If we can take a radius $r > 0$ independent of x then we say that Ω satisfies the uniform sphere condition.

Remark) If $\partial\Omega$ smooth, Ω is bdd. then Ω satisfies the uniform exterior sphere condition.

Thm) C^0-estimate.

Ω is open and bounded in $\mathbb{R}^n$.

and satisfies the uniform exterior sphere condition.

$c(x) \leq 0$. $|b_i(x)| \leq \Lambda$.

$\lambda |\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda |\xi|^2$, $\lambda > 0$.

f is bounded.

suppose $u \in C^0(\bar{\Omega}) \cap D^2(\Omega)$.

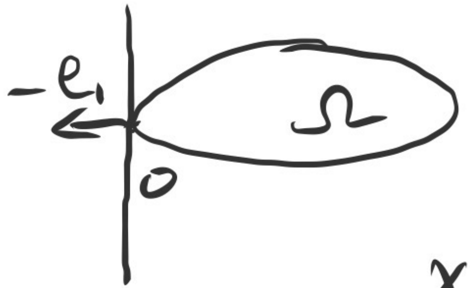
satisfies $f = Lu = a_{ij}u_{ij} + b_i u_i + cu$. in Ω

$u = 0$ on $\partial\Omega$.

Then. $\sup_{\Omega} |u| \leq C \sup_{\Omega} |f|$

where. $C = C(n, \mathrm{diam}(\Omega), \lambda, \Lambda)$

pf) your final assignment!! (cf. midterm) pf for the case Ω is convex)



given $x_0 \in \partial\Omega$.

By rotation and translation

$$x_0 \to 0, \quad \nu = -e_1$$

we construct a barrier

$$\varphi(x) = M(1 - e^{-\alpha x_1}) \sup|f|$$

where $\alpha = 2\Lambda/\lambda$.

$$M = \frac{2}{\Lambda\alpha} e^{\alpha D}, \quad D = \operatorname{diam}(\Omega)$$

Then, $\varphi(x) \geq 0$ in $\overline{\Omega}$.

$$a_{ij}\varphi_{ij} + b_i\varphi_i = a_{11}\varphi_{11} + b_1\varphi_1$$
$$= M(\sup|f|)(-a_{11}\alpha^2 + b_1\alpha)e^{-\alpha x_1}$$
$$\leq (\sup|f|)M\left(-\cancel{\lambda}\cdot\frac{2\Lambda}{\cancel{\lambda}}\alpha + \Lambda\alpha\right)e^{-\alpha D}$$
$$= -(\sup|f|)\cancel{M}\,\cancel{\Lambda\alpha e^{-\alpha D}} \leq -2\sup|f|.$$

We claim that $u \le \varphi$ in $\bar{\Omega}$.

Suppose NOT.

Then. $h = \sup\limits_{\bar{\Omega}} (u - \varphi) > 0$.

and $\exists\, x_0 \in \Omega$ s.t. $u(x_0) - \varphi(x_0) = h$.

At the interior max point x_0

$$0 \geq a_{ij}(u_{ij} - \varphi_{ij}) + b_i(u_i - \varphi_i)$$

$$= (a_{ij}u_{ij} + b_i u_i) - (a_{ij}\varphi_{ij} + b_i \varphi_i)$$

$$\geq f - cu + 2\sup|f|$$

$$\geq -(\sup|f|) + 0 + 2\sup|f| = \sup|f|$$

$\Rightarrow f \equiv 0$ in $\bar{\Omega}$

By max principle. $u \equiv 0$ in $\bar{\Omega}$

$\Rightarrow u \le \varphi$. !!

Similarly. $u \geq -\varphi$. $\Rightarrow |u| \le \varphi \le C\sup|f|$

where $C = C(n, \mathrm{diam}(\Omega), \lambda, \Lambda)$ ∎

Idea for a general domain)

$x_0 = re_1 \in \partial B_r(0) \cap \partial\Omega$

$B_r(0) \cap \Omega = \emptyset$.

Construct $\varphi(x) = \psi(|x|)$

$\varphi = 0$ on $\partial B_r(0)$.

$a_{ij}\varphi_{ij} + b_i\varphi_i \le -2\sup|f|$.

Given open bdd $\Omega \subset \mathbb{R}^n$ w/ C^∞ $\partial\Omega$.

we denote $\mathcal{B} = C^{2,\alpha}(\bar{\Omega})$

$V = C^\alpha(\bar{\Omega})$

$\|\cdot\|_{\mathcal{B}} = \|\cdot\|_{C^{2,\alpha}(\bar{\Omega})}$, $\|\cdot\|_V = \|\cdot\|_{C^\alpha(\bar{\Omega})}$

prop) $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$, $(V, \|\cdot\|_V)$

are complete normed vector space !!

pf) $u, v \in C^{2,\alpha} \Rightarrow au + bv \in C^{2,\alpha}$

for any $a, b \in \mathbb{R}$.

$\Rightarrow \mathcal{B}$ is a "vector space"

$\|u+v\|_{C^{2,\alpha}} \leq \|u\|_{C^{2,\alpha}} + \|v\|_{C^{2,\alpha}}$

(c.f. #1 pset 4)

$\|u\|_{C^{2,\alpha}} \geq 0$, (= hold iff $u=0$)

$\|au\|_{C^{2,\alpha}} = |a| \|u\|_{C^{2,\alpha}}$, $a \in \mathbb{R}$.

$\hookrightarrow \|\cdot\|_{\mathcal{B}}$ is a norm of $\mathcal{B}$

Given a Cauchy seq $\{u_n\} \subset C^{2,\alpha}$

$$\lim_{i,j\to+\infty} \|u_i - u_j\|_{C^{2,\alpha}} = 0,$$

there exists the limit

$$u = \lim_{m\to\infty} u_m \in C^{2,\alpha}(\bar{\Omega})$$

(c.f. #2 in pset 4).

$\Rightarrow (B, \|\cdot\|_B)$ is complete

Similarly $(V, \|\cdot\|_V)$ is complete normed vector space.

Def) A Banach space is a complete normed vector space.

Define $B_0 = \{u \in B \mid u = 0 \text{ on } \partial\Omega\}$

$V_0 = \{u \in V \mid u = 0 \text{ on } \partial\Omega\}$

$\Rightarrow (B_0, \|\cdot\|_B)$, $(V_0, \|\cdot\|_V)$

are Banach spaces.

Remark) $(C^2, \|\cdot\|_{C^2})$ is NOT complete.

~~A~~ A Cauchy seq $u_m \in C^2 (\subset C^{1,1})$

may have the limit

$$u = \lim_{m \to \infty} u_m \in C^{1,1}$$

$$u \notin C^2.$$

c.f. #2 pset 4.

Let $a_{ij}, b_i, c \in C^{\alpha}(\bar{\Omega})$

$$\lambda|\xi|^2 \le a_{ij}\xi_i\xi_j \le \Lambda|\xi|^2$$

$$\|b_i\|_{C^\alpha},\ \|a_{ij}\|_{C^\alpha},\ \|c\|_{C^\alpha} \le \Lambda$$

$$a_{ij} = a_{ji}, \quad c \le 0.$$

Define $L_t = t\Delta + (1-t)L$

for $t \in [0,1]$

i.e. $L_t u = t\Delta u$

$$+ (1-t)(a_{ij}u_{ij} + b_i u_i + cu).$$

$$= \big(t\delta_{ij} + (1-t)a_{ij}\big)u_{ij}$$

$$+ \underbrace{(1-t)b_i}_{b_i^t} u_i + \overbrace{(1-t)c}^{c^t} u.$$

$$\lambda|\xi|^2 \le \underbrace{(t\delta_{ij} + (1-t)a_{ij})}_{a_{ij}^t}\xi_i\xi_j \le \Lambda|\xi|^2$$

$$\|a_{ij}^t\|_{C^\alpha},\ \|b_i^t\|_{C^\alpha},\ \|c^t\|_{C^\alpha} \le \Lambda + 1.$$

Thm) $f \in C^{\alpha}(\bar{\Omega})$, $u \in C^{2,\alpha}(\bar{\Omega})$

$\alpha \in (0,1)$, $t \in [0,1]$

$L_t u = f$ in $\bar{\Omega}$, $u = 0$ on $\partial\Omega$

Then, $\exists\ C = C(n, \alpha, \Omega, \lambda, \Lambda)$ s.t.

$$\|u\|_B = \|u\|_{C^{2,\alpha}} \le C\|f\|_{C^\alpha} = C\|f\|_V$$

pf) By the (global) Schauder est.

$$\|u\|_{C^{2,\alpha}} \le C\left([u]_{0;\bar{\Omega}} + \|f\|_{C^\alpha}\right)$$

By the C^0 estimate,

$$\sup_{\Omega} |u| = [u]_{0;\bar{\Omega}} \le C \sup|f| \le C\|f\|_{C^\alpha}$$

$\Rightarrow \|u\|_{C^{2,\alpha}} \le C\|f\|_{C^\alpha}$. ☒

Thm) $u \in B_0 \Rightarrow \|u\|_B \le C\|L_t u\|_V$

where $C = C(n, \alpha, \Omega, \lambda, \Lambda)$

Error at remark in pp 11

C^k is ~~NOT~~ a Banach space

Fact) C^k IS a Banach space.

c.f) #2-(2) pset 4.

$C^{k,\alpha}$ is a Banach space.

$$A_M = \{u \in C^k(\bar{\Omega}) \mid \|u\|_{C^k} \leq M\}$$

(M is a constant)

Then. A_M is NOT sequentially compact.

Namely, $\exists \{u_i\} \subset A_M$ s.t.

$\{u_i\}$ has no subseq. which is convergent in A_M.

c.f) #2-(1) pset 4

$$B_M = \{ u \in C^{k,\alpha} \mid \|u\|_{C^{k,\alpha}} \le M \}$$
$(\alpha \in (0,1])$

B_M is seq. compact.

c.f. #2-(3) in pset 4.

Application) suppose $a_{ij}^m, b_i^m, c^m, f^m \in C^\alpha$, $g \in C^\infty$

$\|a_{ij}^m\|_{C^\alpha}, \|b_i^m\|_{C^\alpha}, \|c^m\|_{C^\alpha} \le \Lambda$

$a_{ij}^m \xi_i \xi_j \ge \lambda |\xi|^2$, $\lambda > 0$

$\|f^m\|_{C^\alpha}, \|g^m\|_{C^{2,\alpha}} \le C_0$.

$\Rightarrow$ we obtain $u^m \in C^\infty$ s.t $u^m = g^m$ on $\partial\Omega$

$a_{ij}^m u_{ij}^m + b_i^m u_i^m + c^m u^m = f^m$ in Ω.

$\Rightarrow \|u^m\|_{C^{2,\alpha}} \le C_1$ independent of m.

$\Rightarrow \exists$ a subsequential limit $u \in C^{2,\alpha}$ s.t.

$Lu = f$ in Ω, $u = g$ on $\partial\Omega$

$g^m \to g$
$f^m \to f$
$a_{ij}^m \to a_{ij}$